A CERTAIN ALGORITHM OF THE SOLUTION OF NONLINEAR PROBLEMS OF THE THEORY OF SHALLOW SHELLS

(OB ODNOM ALGORIFME RESHENIIA NELINEINYKH ZADACH TEORII POLIGKH OBOLOCHEK)

PMM Vol.23, No.1, 1959, pp. 159-163

M.S. KORNISHIN and Kh.M. MUSHTARI
(Kazan')

(Received 19 May 1958)

This paper gives a description of an algorithm proposed by Mushtari for the solution of systems of nonlinear algebraic equations for nonlinear problems of the theory of shallow shells.

This algorithm may also be applied in the study of finite bending of shallow cylindrical panels of rectangular planform with simply supported edges under the influence of uniformly distributed loading. The problem may be solved by the Bubnov-Galerkin method for the integration of the strain compatibility and equilibrium equations. The expressions selected for the stress and flexure functions satisfy all the static and geometric boundary conditions at every point of the boundary (Section 1 is written by Mushtari, Sections 2-3 by Kornishin). The numerical work has been performed on the electronic computer "Strela" of the Computing Center of the Academy of Sciences of the USSR by means of a program composed by Skvortsov, to whom the authors express their gratirude.

1. A method for the approximate solution of problems of the theory of average bending of elastic shallow shells. The basic equations of the nonlinear theory of very shallow shells of rectangular planform, and also of shallow cylindrical shells, may be reduced to the form

$$D\triangle\triangle w - T_1\left(\frac{\partial^2 w}{\partial x^2} + k_1\right) - 2T_{12}\frac{\partial^2 w}{\partial x\,\partial y} - T_2\left(\frac{\partial^2 w}{\partial y^2} + k_2\right) - p = 0 \qquad (1.1)$$

$$\triangle\triangle\psi = Et\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} + k_2\right) - k_1\frac{\partial^2 w}{\partial y^2}\right] \quad \left(D = \frac{Et^3}{12(1-\nu^2)}\right) \qquad (1.2)$$

Here t is the thickness, E Young's modulus, ν Poisson's ratio of the material of the shell, k_1, k_2 are the principal curvatures of the middle surface before deformation, w is the deflection, assumed positive on the side of the internal normal, p is the static pressure per unit area,

T_1, T_{12}, T_2 are the membrane forces

$$T_1 = \frac{\partial^2\psi}{\partial y^2}, \qquad T_{12} = -\frac{\partial^2\psi}{\partial x\,\partial y}, \qquad T_2 = \frac{\partial^2\psi}{\partial x^2}, \qquad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

For the solution of these nonlinear equations by the method of successive approximations, as a first approximation we apply the solution of the corresponding linear problem, and determine the second approximation by solving the equations obtained from (1.1) and (1.2) after replacement of the nonlinear terms by the values of w and ψ calculated from the first approximation, etc. For this purpose we have to solve repeatedly a system of linear differential equations for given boundary conditions, which in itself presents great difficulties. In addition, there still remains the vague problem of the region of convergence and especially the question of the rate of convergence of the process.

It therefore appeared preferable to integrate equations (1.1) and (1.2) by the Bubnov-Galerkin method, approximating the unknown functions by series of the form

$$w = \sum_{ij} w_{ij} f_{ij}, \qquad \psi = \sum_{ij} \psi_{ij} \varphi_{ij} \tag{1.3}$$

where f_{ij}, ϕ_{ij} are functions of x and y, satisfying the corresponding boundary conditions.

Multiplying (1.1) by f_{ij}, (1.2) by ϕ_{ij} and integrating over the plane of the panel, we obtain a system of algebraic equations to determine amplitudes w_{ij} and ψ_{ij} of the deflections and stress function. Since the ψ_{ij} enter linearly, their expressions may be found in terms of w_{ij} (which, however, is not essential). We thus obtain the system of cubic equations for the w_{ij}

$$\sum_{ij} A_{ij}{}^{m} w_{ij} + F^m(w_{ij}) = B_m p \qquad (m = 1, 2, \ldots) \tag{1.4}$$

where $F^m(w_{ij})$ are the nonlinear parts of the corresponding equations and the $A_{ij}{}^m$ and B_m have known numerical values.

In the general case, for any realistic number of terms of the series (1.3) the solution of this system leads to very involved computations, the execution of which even when using a high-speed computer demands a considerable amount of machine time. In addition, it is extremely laborious to construct the Bubnov-Galerkin equations and program their solution for the machine. Taking these facts into consideration, it was decided to limit the range of problems to be solved to "average bending", when the deflections do not become too large, i.e. not exceeding 1.5 to 2 times the thickness of the shell. In this case, if in the solution of the linear problem only a small number of the amplitudes of the principal harmonics are dominant, and if the remaining harmonics influence the magnitude of the stresses only slightly, so that not only series (1.3) but also the series for $\partial^2 w/\partial x^2$, ... converge well, for the solution of the nonlinear problems satisfactory convergence of these series may be obtained. In paper [1] it has been shown that it is admissible to linearize the stated equations with respect to the nondominant amplitudes and to estimate the accuracy obtained. Here, with the object of expanding the area of applicability and of defining error estimates of a simplified method of solution, equations (1.4) will not be linearized, but the following approximate method of solution will be proposed instead.

To explain the idea of the method, consider a problem in the solution of which one term of the series for w with amplitude w_{11} plays the principal role. Let it be required to construct a table of values of w_{ij} (and also of the deflections and membrane stresses) for a number of values of loading p. Instead, the consecutive values of $|w_{11}|$, beginning with its small values, will be given, for example beginning with $|w_{11}| = 0.1\ t$. For this value of $|w_{11}|$ the magnitudes of the other amplitudes w_{12}, w_{21}, ..., will differ only slightly from the value w_{ij}^L of the linear theory. Therefore, if in the expressions $F^m(w_{ij})$ the term $|w_{11}|$ is replaced by $0.1\ t$ and in the remaining nonlinear terms the w_{ij} are replaced by w_{ij}^L, the errors incurred in the F^m will be very small; since these nonlinear terms are small compared with the linear terms, from the system of linear equations obtained the corresponding quantity p and the remaining $(n-1)$ amplitudes (if one limits consideration to n terms of the series approximating w) are found to a high degree of accuracy. Substituting next in (1.4) $|w_{11}| = 0.2\ t$, and replacing in the nonlinear terms the remaining w_{ij} by their values obtained for $|w_{11}| = 0.1\ t$, a new system of linear equations is found. Solving this system gives the corresponding values of p and w_{ij}. Proceeding in an analogous manner, a table is obtained from which one may find by interpolation the required quantities for given values of the loading.

It is obvious from the above that the error in the w_{ij} quantities will increase with every step. However, it will remain small, because $w_{ij}\partial^2 f_{ij}/\partial x^2$ are not quantities of the same order of magnitude as $w_{11}\partial^2 f_{11}/\partial x^2$, since for this principal term the nonlinear part of the equation (1.4) has been evaluated exactly. Finally, system (1.4) may be solved by means of the usual method of successive approximations: for a given value of p all the w_{ij} may be determined from the linear theory, and the w_{ij}^L values found may be substituted in the F^m; then, solving the linearized sy system for w_{ij}, we find their improved values, etc., until the corrections obtained become negligibly small. However, for this process, the computations are necessarily very cumbersome, since they must be often repeated and then again for every value of p.

The proposed method has the advantage of great simplicity. To estimate the accuracy attained and to reduce the error, this method may be combined with the above method of successive approximation, applying the latter, for example, after every ten steps in order to refine the w_{ij} quantities. The results of the calculations of the last step may also be compared with the results obtained by any other method, which will permit an estimate of the error incurred. Such an estimate is given in the following example of the problem of transverse bending of flexible panels. It shows the slow growth of the admissible error with increasing bending even for a step of 0.25 t. With increasing steplength, the error grows rapidly. The method may also be applied to the solution of other nonlinear problems, such as the study of the behavior of shells after instability has occurred, nonlinear vibrations, etc.

2. Bending of shallow cylindrical panels with freely supported edges. The basic equations for shallow cylindrical panels are obtained from (1.1), (1.2) by letting $k_1 = 0$, $k_2 = 1/R$ where R is the radius of the cylinder. Placing the origin of coordinates at the center of the panel, for the boundary conditions of free support we have

$$\psi_{yy} = \psi_{xy} = 0, \qquad w_{xx} = w = 0 \qquad \text{for } x = \pm a$$
$$\psi_{xx} = \psi_{xy} = 0, \qquad w_{yy} = w = 0 \qquad \text{for } y = \pm b \tag{2.1}$$

where $2a$ and $2b$ denote the length and the width of the panel. The condition (2.1) will be exactly satisfied by letting

$$\psi = \psi_{11}\left(1 + \cos\frac{\pi x}{a}\right)\left(1 + \cos\frac{\pi y}{b}\right) + \psi_{12}\left(1 - \cos\frac{2\pi x}{a}\right)\left(1 + \cos\frac{\pi y}{b}\right) + $$
$$+ \psi_{21}\left(1 + \cos\frac{\pi x}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right) \tag{2.2}$$

$$w = w_{11}\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + w_{12}\cos\frac{\pi x}{2a}\cos 3\frac{\pi y}{2b} + w_{21}\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} \tag{2.3}$$

Substituting (2.2), (2.3) in (1.1), (1.2) and applying to each the Bubnov method, after several transformations we obtain the relations:

$$[2\lambda^4 + 2 + (\lambda^2+1)^2]\, f_{11} + 2f_{12} + 2\lambda^4 f_{21} = -0.0625\,\lambda^2\,(2\zeta_{11}^2 + 9\zeta_{12}^2 +$$
$$+ 9\zeta_{21}^2 + 6\zeta_{11}\zeta_{12} + 6\zeta_{11}\zeta_{21} + 16\zeta_{12}\zeta_{21}) + 0.1643\,k\lambda^2\,(0.1111\,\zeta_{11} + 0.0222\,\zeta_{12} + 0.200\,\zeta_{21})$$
$$2f_{11} + [32\lambda^4 + 2 + (4\lambda^2+1)^2]\, f_{12} = 0.0625\,\lambda^2\,(-\zeta_{11}^2 - 9\zeta_{21}^2 - 2\zeta_{11}\zeta_{12} +$$
$$+ 9\zeta_{11}\zeta_{21} + 25\zeta_{12}\zeta_{21}) + 0.6572\,k\lambda^2\,(0.0222\,\zeta_{11} + 0.0044\,\zeta_{12} - 0.1429\,\zeta_{21})$$
$$2\lambda^4 f_{11} + [2\lambda^4 + 32 + (\lambda^2+4)^2]\, f_{21} = 0.0625\,\lambda^2\,(-\zeta_{11}^2 - 9\zeta_{12}^2 + 9\zeta_{11}\zeta_{12} -$$
$$- 2\zeta_{11}\zeta_{21} + 25\zeta_{12}\zeta_{21}) + 0.6572\,k\lambda^2\,(0.0222\,\zeta_{11} - 0.0159\,\zeta_{12} + 0.040\,\zeta_{21}) \tag{2.4}$$

$$p^* = 5.502\,(\lambda^2+1)^2\zeta_{11} + \lambda^2 k\,(17.543\,f_{11} + 14.035\,f_{12} + 2.807\,f_{21}) -$$
$$-\,240.34\,\lambda^2\,[f_{11}\,(\zeta_{11} + 1.5\,\zeta_{12} + 1.5\,\zeta_{21}) + f_{12}\,(0.5\,\zeta_{11} + 0.5\,\zeta_{12} - 2.25\,\zeta_{21}) +$$
$$+\,f_{21}\,(0.5\,\zeta_{11} + 0.5\,\zeta_{21} - 22.5\,\zeta_{12}) \qquad (2.5)$$

$$\lambda^2\,[f_{11}\,(5.5\,\zeta_{11} + 15\,\zeta_{12} + 13.5\,\zeta_{21}) + f_{12}\,(0.875\,\zeta_{11} + 0.5\,\zeta_{12} - 6.937\,\zeta_{21}) +$$
$$+\,f_{21}\,(-1.187\,\zeta_{11} + 1.125\,\zeta_{12} - 4.187\,\zeta_{21})] - k\lambda^2\,(0.1168\,f_{11} + 0.0934 f_{12} +$$
$$+\,0.1368\,f_{21}) - 0.0229\,(\lambda^2+1)^2\zeta_{11} - 0.0687\,(\lambda^2+9)^2\zeta_{12} = 0 \qquad (2.6)$$

$$\lambda^2\,[f_{11}(5.5\zeta_{11} + 13.5\zeta_1{}^2 + 15\zeta_{21}) + f_{12}\,(-1.187\zeta_{11} - 4.187\zeta_{12} + 1.125\zeta_{21}) +$$
$$+\,f_{21}(0.875\zeta_{11} - 6.937\zeta_{12} + 0.50\zeta_{21}) - k\lambda^2\,(0.4672 f_{11} - 1.0679 f_{12} +$$
$$+\,0.3271 f_{21}) - 0.0229\,(\lambda^2+1)^2\zeta_{11} - 0.0687\,(9\lambda^2+1)^2\zeta_{21} = 0$$

In these equations,

$$f_{ik} = \frac{\psi_{ik}}{Et^3}, \qquad \zeta_{ik} = \frac{w_{ik}}{t}, \qquad k = \frac{4b^2}{Rt}, \qquad p^* = \frac{16\,pb^4}{Et^4}, \qquad \lambda = \frac{b}{a}\,.$$

k is a parameter for the curvature, q^* a parameter for the loading.

Starting from (2.4)-(2.6), for given values of the parameters p^* and the curvatures k and λ we may determine the values of the parameters ζ_{ik}, f_{ik}, which characterise the state of stress and deformation of the panel.

Expressions for the membrane stresses $\sigma_x{}^0$, $\sigma_y{}^0$ are deduced below, together with those for bending moments M_x, M_y and deflection ζ_0 at the center of the panel:

$$\sigma_x{}^\circ = E\,\frac{t^2}{b^2}\,\alpha, \qquad \sigma_y{}^\circ = E\,\frac{t^2}{b^2}\,\beta \qquad (2.7)$$

where

$$\alpha = -2\pi^2 f_{11} + 8\pi^2 f_{21},\ \beta = -2\pi^2\lambda^2 f_{11} + 8\pi^2\lambda^2 f_{12},\ M_x = -D\,\frac{t}{b^2}\,\gamma, \qquad M_y = -D\,\frac{t}{b^2}\,\delta \qquad (2.8)$$

and

$$\gamma = \frac{1}{4}\,\pi^2\,[\zeta_{11}\,(\lambda^2+\nu) + \zeta_{12}\,(\lambda^2+9\nu) + \zeta_{21}\,(9\lambda^2+\nu)].$$
$$\delta = \frac{1}{4}\,\pi^2\,[\zeta_{11}\,(1+\nu\lambda^2) + \zeta_{12}\,(9+\nu\lambda^2) + \zeta_{21}\,(1+9\nu\lambda^2)]$$
$$\zeta_0 = \zeta_{11} + \zeta_{12} + \zeta_{21} \qquad (2.9)$$

3. Numerical results and some deductions. The results of the computations are shown in the graphs (Figs. 1, 2) and Table 1. These graphs show the dependence between load parameter p^* and deflection parameter ζ_0 of the center for $\lambda = 1$ (Fig. 1) and $\lambda = 0.5$ (Fig. 2), while Table 1 gives the values of the coefficients in formulas (2.7), (2.8).

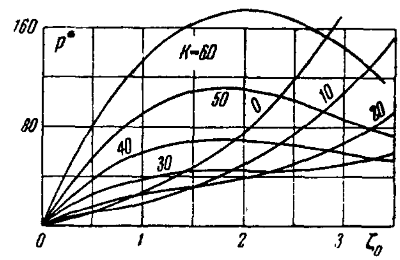

Fig. 1.

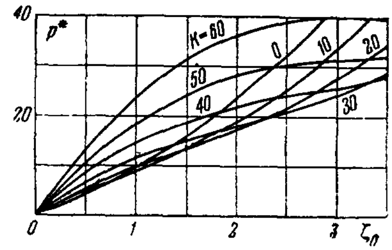

Fig. 2.

For the purpose of the ensuing discussion the following notation will be introduced (having in mind the square plate):

Let p_c^*, α_c, γ_c, ζ_{12}^c be the values of p^*, α, γ, ζ_{12} computed for given ζ_{12} in accordance with the theory of average bending[1].

Let p_1^*, α_1, γ_1, ζ_{12}' be the values of the same quantities computed for the same value of ζ_{11} by use of the above algorithm for $\Delta\zeta_{11} = 0.25$.

Let p_2^*, α_2, γ_2, ζ_{12}'' correspond to the same ζ_{11} with ζ_{12} having been obtained on the basis of ζ_{12}' from the method of successive approximations. These last may be considered to be exact, since ζ_{12}'' has been determined accurately to the fourth decimal place. Finally, let p_0^*, α_0, γ_0 be the solution of the nonlinear problem when only the first term in expression (2.3) for the deflection function is retained, these quantities having been computed for the same value of the deflection at the center of the panel as p_2^*, ..., γ_2. In Table 2, these quantities have been evaluated for the purpose of comparison.

It follows from Table 2 that even for the fairly large steps ($\Delta\zeta_{11} = 0.25$) selected for the algorithm in the present problem, the corresponding values of ζ_{12}' are close to the exact values ζ_{12}'' up to sufficiently large ζ_{11}($\zeta_{11} = 5$). Hence, the algorithm for the solution of the system of nonlinear algebraic equations of Section 1 is very effective. Further, it is clear from a study of the Tables 2 and 3 that if, for the evaluation of deflections of order 2-2.5 times the thickness, the solution obtained by retaining one principal term in the expression for w gives satisfactory results, then the same solution is not quite satisfactory for the evaluation of the stresses.

For deflections of the order of twice the thickness, it is more convenient to apply the theory of average deflection [1], since the correction arising from the complete nonlinear theory is small and requires difficult computations.

$\lambda = 1$

Table 1.

k	ζ_0	p^*	α	β	γ	δ
0	0.484	11.43	0.055	0.055	1.349	1.349
	0.955	25.21	0.204	0.204	2.482	2.482
	1.404	43.10	0.413	0.413	3.281	3.281
	1.832	66.20	0.650	0.650	3.730	3.730
	2.240	95.33	0.891	0.891	3.861	3.861
5	0.487	10.945	—0.014	—0.010	1.406	1.406
	0.971	22.46	0.081	0.092	2.757	2.727
	1.438	36.80	0.257	0.285	3.857	3.773
	1.883	55.52	0.475	0.534	4.624	4.462
	2.307	79.60	0.707	0.812	5.058	4.803
10	0.487	11.47	—0.085	—0.075	1.385	1.411
	0.981	21.52	—0.052	—0.039	2.904	2.900
	1.465	32.79	0.085	0.113	4.304	4.214
	1.930	47.39	0.287	0.353	5.417	5.189
	2.373	66.69	0.513	0.647	6.189	5.783
15	0.483	13.00	—0.154	—0.130	1.290	1.364
	0.983	22.54	—0.190	—0.170	2.904	2.981
	1.485	31.31	—0.100	—0.085	4.579	4.573
	1.972	42.06	0.083	0.120	6.055	5.881
	2.436	56.72	0.311	0.409	7.202	6.784
20	0.475	15.52	—0.218	—0.169	1.134	1.269
	0.978	25.63	—0.329	—0.285	2.760	2.956
	1.495	32.67	—0.296	—0.284	4.656	4.814
	2.005	39.87	—0.135	—0.142	6.500	6.500
	2.493	49.95	0.099	0.119	8.052	7.779
25	0.464	18.93	—0.276	—0.188	0.936	1.132
	0.966	30.78	—0.462	—0.370	2.489	2.825
	1.494	37.09	—0.499	—0.465	4.528	4.907
	2.028	41.18	—0.360	—0.410	6.722	7.005
	2.544	46.73	—0.124	—0.202	8.706	8.734

Table 2.

ζ_{11}	ζ^c_{12}	ζ'_{12}	ζ''_{12}	p^*_c	p^*_1	p^*_2	$\lambda = 1,\ k = 0$	p^*_0	ζ_*
2	—0.0807	—0.0837	—0.0860	64.87	66.20	66.01		63.20	1.828
3	—0.1664	—0.1841	—0.1927	124.12	131.20	129.95		124.79	2.615
5	—0.3621	—0.4660	—0.4991	237.12	356.93	348.03		329.08	4.00

Table 3.

ζ_{11}	α_c	α_1	α_2	γ_c	γ_1	γ_2	$\lambda = 1,\ k = 0$	ζ_*	α_*	γ_*
2	0.66	0.65	0.64	3.74	3.73	3.66		1.828	0.87	5.70
3	1.23	1.13	1.07	4.30	3.72	3.43		2.615	1.75	8.40

BIBLIOGRAPHY

1. Mushtari, Kh. M., Srednii izgib pologoi obolochki, priamougol'noi v vlane i opiraiushcheisia na gibkie v svoei ploskosti rebra (Average bending of shallow shells of rectangular planform resting on ribs flexible in their planes). *Izv. Kazanskogo Filiala AN SSR, Ser. fiz.-mat. tekh. nauk* No. 12, 1958.

Translated by J.R.M.R.